

A REMARK ON GLOBAL WELL-POSEDNESS BELOW L^2 FOR THE GKDV-3 EQUATION

AXEL GRÜNROCK, MAHENDRA PANTHEE, AND JORGE DRUMOND SILVA

ABSTRACT. The I -method in its first version as developed by Colliander et al. in [2] is applied to prove that the Cauchy-problem for the generalised Korteweg-de Vries equation of order three (gKdV-3) is globally well-posed for large real-valued data in the Sobolev space $H^s(\mathbb{R} \rightarrow \mathbb{R})$, provided $s > -\frac{1}{42}$.

1. INTRODUCTION

In a recently published paper of Tao [12] concerning the Cauchy-problem for the generalised Korteweg-de Vries equation of order three (for short: gKdV-3), i.e.:

$$u_t + u_{xxx} \pm (u^4)_x = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, \quad (1)$$

it was shown that this problem is locally well-posed for data u_0 in the critical Sobolev space $\dot{H}^{-\frac{1}{6}}(\mathbb{R} \rightarrow \mathbb{C})$ and globally well-posed for data with sufficiently small $\dot{H}^{-\frac{1}{6}}$ -norm. Moreover, scattering results in $H^1 \cap \dot{H}^{-\frac{1}{6}}(\mathbb{R} \rightarrow \mathbb{R})$ for the radiation component of a perturbed soliton were obtained. Tao's local result improves earlier work of Kenig, Ponce, and Vega ($s \geq \frac{1}{12}$, see [9, Theorem 2.6]) and of the first author ($s > -\frac{1}{6}$, see [7]), while the global *small* data theory seems to be completely new in Sobolev spaces of negative index. For *large real valued* data $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$, $s \geq 0$, global well-posedness of (1) was obtained in [7] by combining the conservation of the L^2 -norm with the local L^2 -result, for $s \geq 1$ this was already in [9, Corollary 2.7], where the energy conservation was used.

Starting with Bourgain's splitting argument [1] and followed by the “ I -method” or “method of almost conservation laws” introduced and further refined by Colliander, Keel, Staffilani, Takaoka, and Tao in a series of papers - see e. g. [2], [3], [4], [5], [6] - effective techniques have been developed, which are capable to show large data global well-posedness *below* certain conserved quantities such as the energy or the L^2 -norm. The question of whether and to what extent these methods apply to the Cauchy-problem for gKdV-3, was raised as well by Linares and Ponce [10, p.177 and p.183] as by Tao, see Remark 5.3 in [12]. In this note we establish global well-posedness of (1) for large data $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$, provided $s > -\frac{1}{42}$, thus giving a partial answer to this question. Our proof combines the first version of the I -method as in [2] with a sharp four-linear $X_{s,b}$ -estimate exhibiting an extra¹ gain of half a derivative.

1991 *Mathematics Subject Classification.* 35Q53.

Key words and phrases. generalised KdV equation of order three – global well-posedness – I -method.

M.P. is partially supported by the Fundação para a Ciência e a Tecnologia through the program POCI 2010/FEDER and by the grant SFRH/BPD/22018/2005.

J.D.S. is partially supported by the Fundação para a Ciência e a Tecnologia through the program POCI 2010/FEDER and by the project POCI/FEDER/MAT/55745/2004.

¹i. e. beyond the cancellation of the derivative in the nonlinearity

Before we turn to the details, let us point out that substantial difficulties appear, if we try to push the analysis further to lower values of s ; by following the construction of a sequence of “modified energies” in [5] we are led already in the second step to a Fourier multiplier, say μ_s , corresponding to M_4 in [5], with a quadratic singularity, and the argument breaks down.² Our fruitless effort in this direction seems to confirm Tao’s remark, that “it is unlikely that these methods would get arbitrarily close to the scaling regularity $s = -\frac{1}{6}$.” [12, Remark 5.3]

Acknowledgement: The first author, A. G., wishes to thank the Center of Mathematical Analysis, Geometry and Dynamical Systems at the IST in Lisbon for its kind hospitality during his visit.

2. A VARIANT OF LOCAL WELL-POSEDNESS, THE DECAY ESTIMATE, AND THE MAIN RESULT

Here we follow the lines of [2]: The operator I_N is defined via the Fourier transform by

$$\widehat{I_N u}(\xi) := m\left(\frac{|\xi|}{N}\right)\widehat{u}(\xi),$$

where $m : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a smooth monotonic function with $m(x) = 1$ for $x \leq 1$ and $m(x) = x^s$, $x \geq 2$. Here $s < 0$, so that $0 < m(x) \leq 1$. $I_N : H^s \rightarrow L^2$ is isomorphic and $\|I_N \cdot\|_{L^2}$ defines an equivalent norm on H^s , with implicit constants depending on N .

The crucial nonlinear estimate in the proof of local well-posedness for (1) with H^s -data, $s > -\frac{1}{6}$, is

$$\|\partial_x \prod_{i=1}^4 u_i\|_{X_{s,b'}} \lesssim \prod_{i=1}^4 \|u_i\|_{X_{s,b}}, \quad (2)$$

which holds true, whenever $0 \geq s > -\frac{1}{6}$, $-\frac{1}{2} < b' < s - \frac{1}{3}$ and $b > \frac{1}{2}$, see [7, Theorem 1]. The $X_{s,b}$ -norms used here are given by

$$\|u\|_{X_{s,b}} = \left(\int d\xi d\tau \langle \tau - \xi^3 \rangle^{2b} \langle \xi \rangle^{2s} |\mathcal{F}u(\xi, \tau)|^2 \right)^{\frac{1}{2}},$$

where \mathcal{F} denotes the Fourier transform in both variables. Later on we shall also use the restriction norms $\|v\|_{X_{s,b}(\delta)} = \inf \{ \|u\|_{X_{s,b}} : u|_{[0,\delta] \times \mathbb{R}} = v \}$. Applying the interpolation lemma [6, Lemma 12.1] to (2) we obtain, under the same assumptions on the parameters s , b' and b ,

$$\|I_N \partial_x \prod_{i=1}^4 u_i\|_{X_{0,b'}} \lesssim \prod_{i=1}^4 \|I_N u_i\|_{X_{0,b}}, \quad (3)$$

where the implicit constant is *independent* of N . Now familiar arguments invoking the contraction mapping principle give the following variant of local well-posedness.

Lemma 1. *For $s > -\frac{1}{6}$ the Cauchy-problem (1) is locally well-posed for data $u_0 \in (H^s, \|I_N \cdot\|_{L^2})$. The lifespan δ of the local solution u satisfies*

$$\delta \gtrsim \|I_N u_0\|_{L^2}^{-\frac{18}{6s+1}} \quad (4)$$

and moreover we have for $b = \frac{1}{2} +$

$$\|I_N u\|_{X_{0,b}(\delta)} \lesssim \|I_N u_0\|_{L^2}. \quad (5)$$

²A similar problem was observed by Tzirakis for the quintic semilinear Schrödinger equation in one dimension, see the concluding remark in [13].

Replacing u^2 by u^4 in the calculation on p. 2 of [2], we obtain for a solution u of (1)

$$\|I_N u(\delta)\|_{L^2}^2 - \|I_N u(0)\|_{L^2}^2 \lesssim \|\partial_x(I_N u^4 - (I_N u)^4)\|_{X_{0,-b}(\delta)} \|I_N u\|_{X_{0,b}(\delta)}. \quad (6)$$

The next section will be devoted to the proof that for $b > \frac{1}{2}$, $0 \geq s \geq -\frac{1}{8}$

$$\|\partial_x(I_N u^4 - (I_N u)^4)\|_{X_{0,-b}(\delta)} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|_{L^2}^4 \quad (7)$$

(see Corollary 1 below), which together with (6) and (5) gives

$$\|I_N u(\delta)\|_{L^2} - \|I_N u(0)\|_{L^2} \lesssim N^{-\frac{1}{2}} \|I_N u(0)\|_{L^2}^4. \quad (8)$$

Now the decay estimate (8) allows us to prove our main result:

Theorem 1. *Let $s > -\frac{1}{42}$ and $u_0 \in H^s(\mathbb{R} \rightarrow \mathbb{R})$. Then the solution u of (1) according to Lemma 1 extends uniquely to any time interval $[0, T]$ and satisfies*

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \lesssim \langle T \rangle^{\frac{-2s}{1+42s}} \|u_0\|_{H^s}. \quad (9)$$

Proof: We choose ε_0 so that Lemma 1 gives the lifespan $\delta = 1$ for all data $\phi \in H^s$ with $\|I_N \phi\|_{L^2} \leq 2\varepsilon_0$. Moreover we demand $16C\varepsilon_0^3 \leq 1$, where C is the implicit constant in the decay estimate (8). Assuming without loss that $T \gg 1$, we fix parameters C_1 , N and λ with

$$2C_1^{-\frac{1}{6}-s} \|u_0\|_{H^s} = \varepsilon_0, \quad N^{\frac{1+42s}{2(1+6s)}} = C_1^3 T, \quad \text{and} \quad \lambda = C_1 N^{\frac{-6s}{1+6s}}.$$

Then $N^{\frac{1}{2}} = \lambda^3 T$ and for $u_0^\lambda(x) = \lambda^{-\frac{2}{3}} u_0(\frac{x}{\lambda})$ it is easily checked that $\|I_N u_0^\lambda\|_{L^2} \leq \varepsilon_0$. For any $k \in \mathbb{N}$, repeated applications of Lemma 1 give a solution u^λ of gKdV-3 with $u^\lambda(0) = u_0^\lambda$ on $[0, k]$, as long as

$$\|I_N u^\lambda(j)\|_{L^2} \leq 2\varepsilon_0, \quad 1 \leq j < k. \quad (10)$$

Since by (8) and the second assumption on ε_0

$$\|I_N u^\lambda(j)\|_{L^2} \leq \varepsilon_0 + jCN^{-\frac{1}{2}}(2\varepsilon_0)^4 \leq \varepsilon_0(1 + jN^{-\frac{1}{2}}),$$

condition (10) is fulfilled for $j \leq N^{\frac{1}{2}} = \lambda^3 T$. Thus u^λ is defined on $[0, \lambda^3 T]$, and with $u(x, t) = \lambda^{\frac{2}{3}} u^\lambda(\lambda x, \lambda^3 t)$ we obtain a solution of (1) on $[0, T]$. Finally we have for $0 \leq t \leq T$

$$\|u(t)\|_{H^s} \lesssim \|I_N u(t)\|_{L^2} \lesssim \lambda^{\frac{1}{6}} \|I_N u^\lambda(\lambda^3 t)\|_{L^2}$$

with $\lambda^{\frac{1}{6}} \sim T^{\frac{-2s}{1+42s}}$ and $\|I_N u^\lambda(\lambda^3 t)\|_{L^2}$ being bounded during the whole iteration process by $2\varepsilon_0 \lesssim \|u_0\|_{H^s}$. This gives the growth bound (9). \square

3. THE DECISIVE FOUR-LINEAR ESTIMATE

Let us first recall several linear and bilinear Airy estimates (in their $X_{s,b}$ -versions), which shall be used below; by interpolation between the sharp version of Kato's smoothing effect (see [8, Theorem 4.1]) and the maximal function estimate from [11, Theorem 3] we have

$$\|J^s u\|_{L_x^p(L_t^q)} \lesssim \|u\|_{X_{0,b}}, \quad (11)$$

whenever $b > \frac{1}{2}$, $-\frac{1}{4} \leq s \leq 1$ and $(\frac{1}{p}, \frac{1}{q}) = (\frac{1-s}{5}, \frac{1+4s}{10})$. We will use (11) with $s = 0$, i.e.

$$\|u\|_{L_x^5(L_t^{10})} \lesssim \|u\|_{X_{0,b}}, \quad (12)$$

and the dual version of (11) with $s = \frac{1}{2}$, which is

$$\|u\|_{X_{\frac{1}{2},-b}} \lesssim \|u\|_{L_x^{\frac{10}{9}}(L_t^{\frac{10}{7}})}. \quad (13)$$

Moreover we shall rely on the Strichartz type estimate

$$\|u\|_{L_{xt}^8} \lesssim \|u\|_{X_{0,b}}, \quad (b > \frac{1}{2}) \quad (14)$$

(cf. [8, Theorem 2.4]) and the bilinear estimate

$$\|I_-^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L_{xt}^2} \lesssim \|u\|_{X_{0,b}} \|v\|_{X_{0,b}}, \quad (b > \frac{1}{2}) \quad (15)$$

from [7, Corollary 1]. Here I^s (J^s) denotes the Riesz (Bessel) potential operator of order $-s$ and I_-^s is defined via the Fourier transform by

$$\widehat{I_-^s(f, g)}(\xi) := \int_{\xi_1 + \xi_2 = \xi} d\xi_1 |\xi_1 - \xi_2|^s \widehat{f}(\xi_1) \widehat{g}(\xi_2).$$

Now we turn to the crucial four-linear $X_{s,b}$ -estimate:

Lemma 2. *Let $b > \frac{1}{2}$, $s_i \leq 0$, $1 \leq i \leq 4$, with $\sum_{i=1}^4 s_i = -\frac{1}{2}$. Then*

$$\|\partial_x \prod_{i=1}^4 v_i\|_{X_{0,-b}} \lesssim \prod_{i=1}^4 \|v_i\|_{X_{s_i,b}}. \quad (16)$$

Proof: We write

$$\|\partial_x \prod_{i=1}^4 v_i\|_{X_{0,-b}} = c \|\xi \langle \tau - \xi^3 \rangle^{-b} \int d\nu \prod_{i=1}^4 \mathcal{F}v_i(\xi_i, \tau_i)\|_{L_{\xi,\tau}^2},$$

where $d\nu = d\xi_1 \dots d\xi_3 d\tau_1 \dots d\tau_3$ and $\sum_{i=1}^4 (\xi_i, \tau_i) = (\xi, \tau)$, and divide the domain of integration into three regions A , B and $C = (A \cup B)^c$. In region A we assume that³ $|\xi_{max}| \leq 1$ and hence $|\xi| \leq 4$, so for this region we get the upper bound

$$\|\prod_{i=1}^4 J^{s_i} v_i\|_{L_{xt}^2} \leq \prod_{i=1}^4 \|J^{s_i} v_i\|_{L_{xt}^8} \lesssim \prod_{i=1}^4 \|v_i\|_{X_{s_i,b}},$$

where in the last step we have used the L_{xt}^8 -Strichartz-type estimate (14). Concerning the region B we shall assume - besides $|\xi_{max}| \geq 1$, implying $\langle \xi_{max} \rangle \lesssim |\xi_{max}|$ - that

- i) $|\xi_{min}| \leq 0.99|\xi_{max}|$ or
- ii) $|\xi_{min}| > 0.99|\xi_{max}|$, and there are exactly two indices $i \in \{1, 2, 3, 4\}$ with $\xi_i > 0$.

Then the region B can be split further into a finite number of subregions, so that for any of these subregions there exists a permutation π of $\{1, 2, 3, 4\}$ with

$$|\xi| \lesssim |\xi|^{\frac{1}{2}} |\xi_{\pi(1)} + \xi_{\pi(2)}|^{\frac{1}{2}} |\xi_{\pi(1)} - \xi_{\pi(2)}|^{\frac{1}{2}} \prod_{i=1}^4 \langle \xi_i \rangle^{s_i}.$$

Assume $\pi = id$ for the sake of simplicity now. Then we get the upper bound

$$\begin{aligned} & \| (I_-^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \|_{X_{\frac{1}{2},-b}} \\ & \lesssim \| (I_-^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1} v_1, J^{s_2} v_2))(J^{s_3} v_3)(J^{s_4} v_4) \|_{L_x^{\frac{10}{9}}(L_t^{\frac{10}{7}})} \\ & \lesssim \| I_-^{\frac{1}{2}} I_-^{\frac{1}{2}}(J^{s_1} v_1, J^{s_2} v_2) \|_{L_{xt}^2} \| J^{s_3} v_3 \|_{L_x^5(L_t^{10})} \| J^{s_4} v_4 \|_{L_x^5(L_t^{10})} \lesssim \prod_{i=1}^4 \|v_i\|_{X_{s_i,b}}. \end{aligned}$$

Here we have applied the estimates (13), Hölder, (15) and (12). Finally we consider the remaining region C : Here the $|\xi_i|$, $1 \leq i \leq 4$, are all very close together and $\gtrsim \langle \xi_i \rangle$. Moreover, at least three of the variables ξ_i have the same sign. Thus for

³Here ξ_{max} is defined by $|\xi_{max}| = \max_{i=1}^4 |\xi_i|$, similarly ξ_{min} .

the quantity *c.q.* controlled by the expressions $\langle \tau - \xi^3 \rangle$, $\langle \tau_i - \xi_i^3 \rangle$, $1 \leq i \leq 4$, we have in this region:

$$c.q. := |\xi^3 - \sum_{i=1}^4 \xi_i^3| \gtrsim \sum_{i=1}^4 \langle \xi_i \rangle^3 \gtrsim \langle \xi \rangle^3.$$

So the contribution of the subregion, where $\langle \tau - \xi^3 \rangle \geq \max_{i=1}^4 \langle \tau_i - \xi_i^3 \rangle$, is bounded by

$$\| \prod_{i=1}^4 J^{s_i} v_i \|_{L_{xt}^2} \leq \prod_{i=1}^4 \| J^{s_i} v_i \|_{L_{xt}^8} \lesssim \prod_{i=1}^4 \| v_i \|_{X_{s_i, b}},$$

where (14) was used again. On the other hand, if $\langle \tau_1 - \xi_1^3 \rangle$ is dominant, we write $\Lambda^{\frac{1}{2}} = \mathcal{F}^{-1} \langle \tau - \xi^3 \rangle^{\frac{1}{2}} \mathcal{F}$ and obtain the upper bound

$$\begin{aligned} \| (\Lambda^{\frac{1}{2}} J^{s_1} v_1) \prod_{i=2}^4 J^{s_i} v_i \|_{X_{0, -b}} &\lesssim \| (\Lambda^{\frac{1}{2}} J^{s_1} v_1) \prod_{i=2}^4 J^{s_i} v_i \|_{L_{xt}^{\frac{8}{7}}} \\ &\leq \| \Lambda^{\frac{1}{2}} J^{s_1} v_1 \|_{L_{xt}^2} \prod_{i=2}^4 \| J^{s_i} v_i \|_{L_{xt}^8} \lesssim \prod_{i=1}^4 \| v_i \|_{X_{s_i, b}}. \end{aligned}$$

Here the dual version $X_{0, -b} \supset L_{xt}^{\frac{8}{7}}$ of the L_{xt}^8 estimate was used first, followed by Hölder's inequality and the estimate itself. The remaining subregions, where $\langle \tau_k - \xi_k^3 \rangle$, $2 \leq k \leq 4$, are maximal, can be treated in precisely the same manner. \square

Corollary 1. *Let $b > \frac{1}{2}$ and $0 \geq s \geq -\frac{1}{8}$. Then*

$$\| \partial_x (I_N (\prod_{i=1}^4 u_i) - \prod_{i=1}^4 I_N u_i) \|_{X_{0, -b}(\delta)} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \| I_N u_i \|_{X_{0, b}(\delta)}. \quad (17)$$

Epecially, if u is a solution of gKdV-3 according to Lemma 1 with $u(0) = u_0$, then

$$\| \partial_x (I_N u^4 - (I_N u)^4) \|_{X_{0, -b}(\delta)} \lesssim N^{-\frac{1}{2}} \| I_N u_0 \|_{L^2}^4. \quad (18)$$

Proof: By (5) the estimate (17) implies (18). Thus it suffices to show

$$\| \partial_x (I_N (\prod_{i=1}^4 u_i) - \prod_{i=1}^4 I_N u_i) \|_{X_{0, -b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \| I_N u_i \|_{X_{0, b}}. \quad (19)$$

Now let ξ_i denote the frequencies of the u_i , $1 \leq i \leq 4$. If all the $|\xi_i| \leq N$, then either $|\xi| \leq N$, such that there's no contribution at all, or we have $|\xi| \geq N$, so that at least, say, $|\xi_1| \geq \frac{N}{4}$. In this case, by Lemma 2, the norm on the left of (19) is bounded by

$$\| \partial_x \prod_{i=1}^4 u_i \|_{X_{0, -b}} \lesssim \| u_1 \|_{X_{-\frac{1}{2}, b}} \prod_{i=2}^4 \| u_i \|_{X_{0, b}} \lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \| I_N u_i \|_{X_{0, b}}.$$

Otherwise there are k large frequencies for some $1 \leq k \leq 4$. By symmetry we may assume that $|\xi_1|, \dots, |\xi_k| \geq N$ and $|\xi_{k+1}|, \dots, |\xi_4| \leq N$. Then we have, again by Lemma 2,

$$\begin{aligned} \| \partial_x \prod_{i=1}^4 I_N u_i \|_{X_{0, -b}} &\lesssim \prod_{i=1}^k \| I_N u_i \|_{X_{-\frac{1}{2k}, b}} \prod_{i=k+1}^4 \| I_N u_i \|_{X_{0, b}} \\ &\lesssim N^{-\frac{1}{2}} \prod_{i=1}^4 \| I_N u_i \|_{X_{0, b}} \end{aligned}$$

as well as

$$\|\partial_x I_N \prod_{i=1}^4 u_i\|_{X_{0,-b}} \lesssim \prod_{i=1}^k \|u_i\|_{X_{-\frac{1}{2k},b}} \prod_{i=k+1}^4 \|u_i\|_{X_{0,b}}.$$

Since for any $s_1 \leq s$ and any v with frequency $|\xi| \geq N$ it holds that

$$\|v\|_{X_{s_1,b}} \lesssim N^{s_1-s} \|v\|_{X_{s,b}} \sim N^{s_1} \|I_N v\|_{X_{0,b}},$$

the latter is again bounded by the right hand side of (19). \square

REFERENCES

- [1] Bourgain, J.: Refinements of Strichartz' inequality and Applications to 2D-NLS with critical Nonlinearity, International Mathematics Research Notices 1998, No. 5, 253 - 283
- [2] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global well-posedness for KdV in Sobolev spaces of negative index, Electron. J. Differential Equations 2001, No. 26, 1-7
- [3] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal. 33 (2001), no. 3, 649-669
- [4] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: A refined global well-posedness result for Schrödinger equations with derivative, SIAM J. Math. Anal. 34 (2002), no. 1, 64-86
- [5] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Sharp global well-posedness results for periodic and nonperiodic KdV and modified KdV on \mathbb{R} and \mathbb{T} , J. Amer. Math. Soc. 16 (2003), 705-749
- [6] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Multilinear estimates for periodic KdV equations, and applications, J. Funct. Anal. 211 (2004), no. 1, 173-218
- [7] Grünrock, A.: A bilinear Airy-estimate with application to gKdV-3, Differential Integral equations 18 (2005), no. 12, 1333-1339
- [8] Kenig, C. E., Ponce, G., Vega, L.: Oscillatory Integrals and Regularity of Dispersive Equations, Indiana Univ. Math. J. 40 (1991), 33 - 69
- [9] Kenig, C., Ponce, G., Vega, L.: Wellposedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, CPAM 46 (1993), 527 - 620
- [10] Linares, F., Ponce, G.: Introduction to nonlinear dispersive equations, Publicações Matemáticas, IMPA, 2004
- [11] Sjölin, P.: Regularity of solutions to the Schrödinger equation, Duke Math. J. 55 (1987), no. 3, 699-715
- [12] Tao, T.: Scattering for the quartic generalised Korteweg-de Vries equation, J. Differential Equations 232 (2007), no. 2, 623-651
- [13] Tzirakis, N.: The Cauchy problem for the semilinear quintic Schrödinger equation in one dimension, Differential Integral Equations 18 (2005), no. 8, 947-960

AXEL GRÜNROCK: BERGISCHE UNIVERSITÄT WUPPERTAL, FACHBEREICH C: MATHEMATIK / NATURWISSENSCHAFTEN, GAUSSSTRASSE 20, 42097 WUPPERTAL, GERMANY.

E-mail address: axel.gruenrock@math.uni-wuppertal.de

MAHENDRA PANTHEE: CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS, DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, 1049-001 LISBOA, PORTUGAL.

E-mail address: mpanthee@math.ist.utl.pt

JORGE DRUMOND SILVA: CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS, DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, 1049-001 LISBOA, PORTUGAL.

E-mail address: jsilva@math.ist.utl.pt